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Difference Posets in the Quantum Structures Background[†]

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In this paper the conditions for D-posets to become orthoalgebras, orthomodular posets, orthomodular lattices, MV-algebras, and Boolean algebras are presented. Also some properties of observables are investigated. It is proved that any two regular observables in an atomic σ-complete Boolean D-poset have a joint observable.

1. INTRODUCTION

Based on the study of unsharp logics, fuzzy systems, and quantum mechanical systems, new algebraic structures have been proposed as their models. In weak orthoalgebras introduced by Giuntini and Greuling [10] and in effect algebras introduced by Foulis and Bennet [8] a primary operation is a partially defined sum. In difference posets (D-posets) defined by Kôpka and Chovanec [12] a primary operation is a partially defined difference.

In the first part of this paper we describe some algebraic structures in terms of difference posets. We will study some properties of D-homomorphisms, especially observables, and we will characterize their ranges from the point of view of substructures of D-posets. Finally, we prove that any two observables in an atomic σ-complete Boolean D-poset have a joint observable.

2. D-POSETS AND SOME ALGEBRAIC STRUCTURES

Let $(\mathcal{P}, \leq)$ be a nonempty partially ordered set (poset). Let $\ominus$ be a partial binary operation on $\mathcal{P}$ such that $b \ominus a$ is defined if and only if $a \leq$

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b. Then $\ominus$ is called a *difference* on $\mathcal{P}$ if the following conditions are satisfied:

(D1) $b \ominus a \leq b$.
(D2) $b \ominus (b \ominus a) = a$.
(D3) If $a \leq b \leq c$, then $c \ominus b \leq c \ominus a$ and $(c \ominus a) \ominus (c \ominus b) = b \ominus a$.

In this article we investigate the bounded posets. A poset $(\mathcal{P}, \leq)$ that possesses the greatest element $1_{\mathcal{P}}$ with a difference on $\mathcal{P}$ is said to be a D-poset (difference poset). The properties of D-posets are treated in many articles. See, for example, refs. 3 and 12.

In the following we need the next assertion.

Proposition 2.1. Let $\mathcal{P}$ be a D-poset. Let $a, b, c, \in \mathcal{P}$, $a, \leq c$, $b \leq c$. If there exists a supremum $a \vee b$ in $\mathcal{P}$, then there exists the infimum $(c \ominus a) \wedge (c \ominus b)$ in $\mathcal{P}$ and $c \ominus (a \vee b) = (c \ominus a) \wedge (c \ominus b)$.

Proof. The inequalities $a \leq a \vee b \leq c$ and $b \leq a \vee b \leq c$ imply

$$c \ominus (a \vee b) \leq c \ominus a, \qquad c \ominus (a \vee b) \leq c \ominus b$$

Let $d \in \mathcal{P}$, $d \leq c \ominus a$, $d \leq c \ominus b$. Then

$$a = c \ominus (c \ominus a) \leq c \ominus d, \qquad b = c \ominus (c \ominus b) \leq c \ominus d$$

Therefore $a \vee b \leq c \ominus d \leq c$ and $d \leq c \ominus (a \vee b)$. ■

An orthoalgebra $\mathbb{O}$ [7, 9] is a set containing two special elements 0, 1 and equipped with a partially defined binary operation $\oplus$ subject to the following conditions for all $a, b, c \in \mathbb{O}$:

(OA1) If $a \oplus b$ is defined, then $b \oplus a$ is defined and $a \oplus b = b \oplus a$ (commutativity).
(OA2) If $b \oplus c$ is defined and $a \oplus (b \oplus c)$ is defined, then $a \oplus b$ is defined, $(a \oplus b) \oplus c$ is defined, and $a \oplus (b \oplus c) = (a \oplus b) \oplus c$ (associativity).
(OA3) For any $a \in \mathbb{O}$ there exists a unique $b \in \mathbb{O}$ such that $a \oplus b$ is defined and $a \oplus b = 1$ (orthocomplementation).
(OA4) If $a \oplus a$ is defined, then $a = 0$ (consistency).

The partial ordering on $\mathbb{O}$ is defined as follows: $a \leq b$ if and only if there exists $c \in \mathbb{O}$ such that $a \oplus c$ is defined and $a \oplus c = b$. Then $0 \leq a \leq 1$ holds for all $a \in \mathbb{O}$. The unique element $b \in \mathbb{O}$ such that $a \oplus b = 1$ is denoted by a' and is called the orthocomplement of a; moreover, $a \oplus b$ is defined if and only if $a \leq b'$.

The difference on an orthoalgebra is defined by

$$b \ominus a = (a \oplus b')' \qquad \text{for} \quad a, b \in \mathbb{O}, \quad a \leq b$$

and thus every orthoalgebra forms a D-poset [12]. Even though we can define a partial binary operation $\oplus$ on any D-poset $\mathcal{P}$ by

$$a \oplus b =: 1_{\mathcal{P}} \ominus ((1_{\mathcal{P}} \ominus a) \ominus b) = 1_{\mathcal{P}} \ominus ((1_{\mathcal{P}} \ominus b) \ominus a)$$

for $a, b \in \mathcal{P}$, $a \leq 1_{\mathcal{P}} \ominus b$, $\mathcal{P}$ need not be an orthoalgebra.

We say that a D-poset is regular [14] if it satisfies the following condition:

(R) If $a \in \mathcal{P}$ and $a \leq 1_{\mathcal{P}} \ominus a$, then $a = 0_{\mathcal{P}}$.

The condition (R) of a D-poset implies the following assertion.

Proposition 2.2. Let $\mathcal{P}$ be a regular D-poset. Then $a \vee a' = 1_{\mathcal{P}}$ and $a \wedge a' = 0_{\mathcal{P}}$ for all $a \in \mathcal{P}$, where $a' := 1_{\mathcal{P}} \ominus a$.

Proof. Let $a, a' \leq c$. Put $d = c \ominus a$ and $b = a' \ominus d$. Then $d \leq a' \leq c$ and $b \leq a' \leq c$. Now, $b = (1_{\mathcal{P}} \ominus a) \ominus (c \ominus a) = 1_{\mathcal{P}} \ominus c \leq 1_{\mathcal{P}} \ominus b$. From the regularity it follows that $b = 0_{\mathcal{P}}$. Thus $d = a' = 1_{\mathcal{P}} \ominus a$, i.e., $c \ominus a = 1_{\mathcal{P}} \ominus a$, which implies $c = 1_{\mathcal{P}}$. Finally, we have $0_{\mathcal{P}} = 1'_{\mathcal{P}} = (a \vee a')' = a' \wedge a$. ■

The corollary of the previous proposition is the following assertion: If a D-poset $\mathcal{P}$ is regular and $a \leq b'$ (which is equivalent to $b \leq a'$), then $a \wedge b = 0_{\mathcal{P}}$.

From the above, the following theorem is true:

Theorem 2.3. A D-poset $\mathcal{P}$ is an orthoalgebra if and only if $\mathcal{P}$ is a regular D-poset.

An orthomodular poset [15, 18] is a partially ordered set $(\mathcal{L}, \leq)$ with the least and greatest elements 0, 1, endowed with a unary operation $'$, so-called orthocomplementation, such that:

(OMP1) $a'' = a$ for any $a \in \mathcal{L}$.
(OMP2) $a \leq b$ implies $b' \leq a'$.
(OMP3) $a \leq b'$ implies $a \vee b \in \mathcal{L}$.
(OMP4) $a \vee a' = 1$ for any $a \in \mathcal{L}$.
(OMP5) $a \leq b$ implies $b = a \vee (b \wedge a')$.

Any orthomodular poset can be regarded as an orthoalgebra by defining $a \oplus b := a \vee b$, precisely, in the case $a \leq b$. Conversely, an orthoalgebra $\mathbb{O}$ is an orthomodular poset if $a \vee b \in \mathbb{O}$ whenever $a \leq b'$ and $a \oplus b = a \vee b$ [9].

Any orthomodular poset can be organized as a D-poset, defining $b \ominus a = b \wedge a'$, for $a \leq b$. Thus a regular D-poset $\mathcal{P}$ is an orthomodular poset if and only if for every $a, b \in \mathcal{P}$, $a \leq b'$, their supremum $a \vee b$ exists in $\mathcal{P}$.

Now we give a sufficient and necessary condition for the difference operation on a D-poset $\mathcal{P}$ to become an orthomodular poset.

Theorem 2.4. A D-poset $\mathcal{P}$ is an orthomodular poset if and only if the following condition is satisfied.

(SR) If $a, b \in \mathcal{P}$ such that $a \leq b'$ and $a, b \leq c$, then $b \leq c \ominus a$ (or equivalently $a \leq c \ominus b$).

Proof. Let $\mathcal{P}$ be an orthomodular poset. If $a, b, c \in \mathcal{P}$ such that $a \leq b'$ and $a, b \leq c$, then from the orthomodularity we have $b' = a \vee (b' \wedge a')$. Then $b = a' \wedge (a \vee b) \leq a' \wedge c = c \ominus a$.

It is easy to see that the regularity follows from the condition (SR). Let $a \leq 1_{\mathcal{P}} \ominus b$. Then $a \oplus b = 1_{\mathcal{P}} \ominus ((1_{\mathcal{P}} \ominus b) \ominus a)$ is an upper bound of the elements a, b. If $a, b \leq c$, then by (SR), $b \leq c \ominus a$, and so

$$(1_{\mathcal{P}} \ominus c) = ((1_{\mathcal{P}} \ominus a) \ominus (c \ominus a)) = (1_{\mathcal{P}} \ominus (c \ominus a)) \ominus a \leq (1_{\mathcal{P}} \ominus b) \ominus a$$

Therefore,

$$1_{\mathcal{P}} \ominus ((1_{\mathcal{P}} \ominus b) \ominus a) \leq 1_{\mathcal{P}} \ominus (1_{\mathcal{P}} \ominus c) = c$$

Thus $a \oplus b = a \vee b$. ■

A D-poset $\mathcal{P}$ which is also a lattice with respect to the order relation $\leq$ is called a D-lattice. Then there is a total binary operation $-$ on $\mathcal{P}$, $b - a =: b \ominus (a \wedge b)$, such that the following properties hold:

(DL1) $a - 0_{\mathcal{P}} = a$ for any $a \in \mathcal{P}$.
(DL2) $a, b \in \mathcal{P}$, $a \leq b$ implies $c - b \leq c - a$ for any $c \in \mathcal{P}$.
(DL3) $a - (a - b) = b - (b - a)$ for every $a, b \in \mathcal{P}$.
(DL4) $a \leq b \leq c$ implies $(c - a) - (c - b) = b - a$.

Conversely, if $\mathcal{P}$ is a poset with the least element $0_{\mathcal{P}}$ and the greatest element $1_{\mathcal{P}}$ and $-$ is a binary operation on $\mathcal{P}$ with the properties (DL1)–(DL4), then $\mathcal{P}$ is a D-lattice [3].

Orthomodular poset $(\mathcal{L}, \leq, 0, 1, ')$ is called an orthomodular lattice if it is a lattice with respect to $\leq$.

Any orthomodular lattice $\mathcal{L}$ is a D-lattice where $b - a = b \wedge (a \wedge b)'$ for every $a, b \in \mathcal{L}$. A D-lattice is not an orthomodular lattice in general.

Theorem 2.5. A D-lattice is an orthomodular lattice if and only if it is regular.

Proof. It is clear that every orthomodular lattice is a regular D-lattice.

Let a D-lattice $\mathcal{P}$ be regular. It suffices to prove that for $a, b \in \mathcal{P}$, $a \leq b'$, $a + b := 1_{\mathcal{P}} - ((1_{\mathcal{P}} - b) - a) = a \vee b$.

Evidently $1_{\mathcal{P}} - ((1_{\mathcal{P}} - b) - a) \geq a \vee b \geq a, b$. By Proposition 2.1 we have

$$\begin{aligned}(1_{\mathcal{P}} &- ((1_{\mathcal{P}} - b) - a)) - (a \vee b)\\ &= ((1_{\mathcal{P}} - ((1_{\mathcal{P}} - b) - a)) - b) \wedge ((1_{\mathcal{P}} - ((1_{\mathcal{P}} - a) - b)) - a)\\ &= ((1_{\mathcal{P}} - b) - ((1_{\mathcal{P}} - b) - a)) \wedge ((1_{\mathcal{P}} - a) - ((1_{\mathcal{P}} - a) - b))\\ &= a \wedge b\end{aligned}$$

Then (by Proposition 2.2) $a + b = 1_{\mathcal{P}} - ((1_{\mathcal{P}} - b) - a) = a \vee b$. ∎

A poset $\mathcal{P}$ with the least element $0_{\mathcal{P}}$ and the greatest element $1_{\mathcal{P}}$ is said to be a Boolean D-poset if there is a binary operation $-$ on $\mathcal{P}$ satisfying the conditions (DL1)–(DL3) and the following condition:

(BD4) $(c - b) - a = (c - a) - b$ for every $a, b, c \in \mathcal{P}$.

We say that two elements a, b of a D-poset $\mathcal{P}$ are compatible, and write $a \leftrightarrow b$, if there exist elements $c, d \in \mathcal{P}$, $d \leq a \leq c$, $d \leq b \leq c$, such that $c \ominus a = b \ominus d$ (or equivalently $c \ominus b = a \ominus d$). This notion of compatibility is equivalent to the standard definitions of compatibility in the orthomodular posets. We can characterize every Boolean D-poset as a D-lattice of pairwise compatible elements. Also, we know that every Boolean D-poset is an MV-algebra and, conversely, every MV-algebra is a Boolean D-poset [3].

Theorem 2.6. A Boolean D-poset $\mathcal{P}$ is a Boolean algebra if and only if $\mathcal{P}$ is regular.

Proof. The proof of the previous assertion follows from the fact that in a regular MV-algebra, $a \vee a' = 1$ for every $a \in \mathcal{P}$ [1]. ∎

3. D-HOMOMORPHISMS OF D-POSETS

A nonempty subset $\mathcal{S}$ of a D-poset $\mathcal{P}$ is said to be a *sub-D-poset* of $\mathcal{P}$ if:

(S1) $1_{\mathcal{P}} \in \mathcal{S}$.
(S2) $a, b \in \mathcal{S}$, $a \leq b$, implies $b \ominus a \in \mathcal{S}$.

A sub-D-poset $\mathcal{S}$ of a D-poset $\mathcal{P}$ is a *sub-D-lattice* of $\mathcal{P}$ if, moreover, the following holds:

(S3) The supremum $a \vee b$ and the infimum $a \wedge b$ exist in $\mathcal{S}$ whenever $a, b \in S$.

A sub-D-lattice $\mathcal{S}$ of $\mathcal{P}$ is a *Boolean sub*(-σ-)*algebra* of $\mathcal{P}$ if $\mathcal{S}$ is the Boolean (σ-) algebra (in the sense of Sikorski [17]) with respect to the lattice operations $\vee$ and $\wedge$, and the unary operation $'$: $a \mapsto a' := 1_{\mathcal{P}} \ominus a$.

A sub-D-poset $\mathcal{S}$ of $\mathcal{P}$ is a *Boolean sub-D-poset* (or in other terms an *MV-subalgebra*) of $\mathcal{P}$ if there exists an extension of the partial binary operation $\ominus$ on $\mathcal{S}$ (denoted by $-$) with properties (DL1)–(DL3) and (BD4).

Let $\mathcal{P}$ and $\mathcal{T}$ be two D-posets. A mapping $w: \mathcal{P} \to \mathcal{T}$ is said to be a *D-homomorphism* (of D-posets) if:

(DH1) $w(1_{\mathcal{P}}) = 1_{\mathcal{T}}$.
(DH2) If $a, b \in \mathcal{P}$, $a \leq b$, then $w(a) \leq w(b)$ and $w(b \ominus a) = w(b) \ominus w(a)$.

A D-homomorphism $w: \mathcal{P} \to \mathcal{T}$ is called a σ-*D-homomorphism* if, moreover, the following holds:

(DH3) $(a_n)_{n=1}^{\infty} \subseteq \mathcal{P}$, $a_n \nearrow a$, and $a \in P$ (i.e., $a_n \leq a_{n+1}$ for any $n \in \mathbb{N}$ and $a = \vee_{n=1}^{\infty} a_n$) implies $w(a_n) \nearrow w(a)$.

The following properties result directly from the definition of a D-homomorphism w:

(i) $w(0_{\mathcal{P}}) = 0_{\mathcal{T}}$.
(ii) $w(a') = (w(a))'$ for any $a \in \mathcal{P}$.
(iii) If $a, b \in \mathcal{P}$, $a \leq b'$, then $w(a \oplus b) = w(a) \oplus w(b)$.
(iv) If $a, b \in \mathcal{P}$, $a \leq b$, then $w(b) = w(a) \oplus (w(b) \ominus w(a))$.
(v) If $a, b \in \mathcal{P}$, $a \leftrightarrow b$, then $w(a) \leftrightarrow w(b)$.

If $\mathcal{P}$ and $\mathcal{T}$ are Boolean algebras (or orthomodular posets = quantum logics), then a D-homomorphism from $\mathcal{P}$ to $\mathcal{T}$ is the same thing as a homomorphism of Boolean algebras well known from the classical Boolean algebras theory (or as a homomorphism of logics known from the quantum logics theory). But a D-homomorphism of MV-algebras is not the same mapping as a homomorphism of MV-algebras from the many-valued logics theory. Indeed, a homomorphism of MV-algebras preserves the binary operation of the sum of elements, while a D-homomorphism respects only the orthogonal sum of elements.

The basic notions of the quantum theory are a state (probability measure) and an observable (a quantum paraphrase of a random variable). We can define these notions as D-homomorphisms of special D-posets.

A *state* on $\mathcal{P}$ is a σ-D-homomorphism from a D-poset $\mathcal{P}$ to the unit interval [0, 1] with the usual difference of reals.

A σ-D-homomorphism x from the σ-algebra $\mathcal{B}(\mathbb{R})$ of all Borel subsets of the real line $\mathbb{R}$ to a D-poset $\mathcal{P}$ is called an *observable* (in $\mathcal{P}$).

Now we will investigate in more details some properties of observables in D-posets.

If x is an observable, then there exists the least closed subset $\sigma(x)$ (called the *spectrum* of x), such that $x(\sigma(x)) = 1_{\mathcal{P}}$.

An observable x is said to be (i) *simple* if $\sigma(x)$ is a finite set, and (ii) *discrete* if $\sigma(x)$ is a countable set.

If $\sigma(x) = \{t_1, \ldots, t_n\}$, then $x(\{t_i\}) > 0_{\mathcal{P}}$ for any $i = 1,2, \ldots, n$.

Suppose now that $a_1 > 0_{\mathcal{P}}, \ldots, a_n > 0_{\mathcal{P}}$ are elements of $\mathcal{P}$ such that $\bigoplus_{i=1}^n a_i = 1_{\mathcal{P}}$ and let $t_1, \ldots, t_n$ be real different numbers. Define x by

$$x(E) = \oplus \{a_i: t_i \in E\}, \qquad E \in \mathcal{B}(\mathbb{R})$$

Then x is a simple observable such that $\sigma(x) = \{t_1, \ldots, t_n\}$.

The set $\mathcal{R}(x) = \{x(E): E \in \mathcal{B}(\mathbb{R})\}$ is said to be the *range* of an observable x. We note that if x is an observable in a σ-orthomodular poset $\mathcal{L}$, then the range $\mathcal{R}(x)$ is always a Boolean sub-σ-algebra of $\mathcal{L}$. But the range of an observable in a D-poset is not a sub-D-poset, in general.

Let x and y be two observables in a D-poset $\mathcal{P}$. We say that x is *representable* by y (or x is y-representable) if there exists a Borel measurable function $f: \mathbb{R} \to \mathbb{R}$ such that $x(E) = y(f^{-1}(E))$ for any $E \in \mathcal{B}(\mathbb{R})$.

The following necessary and sufficient conditions for the representation of observables have been proved by Kôpka and Chovanec [12].

Theorem 3.1 (Representation Theorem). An observable x is representable by an observable y if and only if there exists a chain $\mathcal{M}$, $\mathcal{M} \subseteq \mathcal{B}(\mathbb{R})$, such that

$$\{x((-\infty, r)): r \in \mathbb{Q}\} \subseteq \{y(A): A \in \mathcal{M}\}$$

where $\mathbb{Q}$ is the set of all rationals.

Let x be an observable in a D-poset $\mathcal{P}$, and $\mathcal{Q}$ be a nonempty subset of the range $\mathcal{R}(x)$. We say that the observable x has a *V-property* on $\mathcal{Q}$ if for every two Borel sets A, B such that $A \subseteq B$ and for every element $c \in \mathcal{Q}$ such that $x(A) \leq c \leq x(B)$ there exists a Borel set C such that $x(C) = c$ and $A \subseteq C \subseteq B$.

Now we give a sufficient condition for the representation of observables.

Proposition 3.2 [2]. Let x and y be two observables such that the following conditions hold:

(i) $\mathcal{R}(x) \subseteq \mathcal{R}(y)$.

(ii) The observable y has the V-property on $\mathcal{R}(x)$.

Then the observable x is y-representable.

Proposition 3.3. If an observable x in a D-poset $\mathcal{P}$ has the V-property on its range $\mathcal{R}(x)$, then $\mathcal{R}(x)$ is a sub-D-poset of $\mathcal{P}$. Moreover, if $\mathcal{R}(x)$ is a lattice, then $\mathcal{R}(x)$ is a Boolean sub-D-poset (MV-subalgebra) of $\mathcal{P}$.

Proof. It is clear that $1_{\mathcal{P}} = x(\mathbb{R}) \in \mathcal{R}(x)$.

Let $x(E), x(F) \in \mathcal{R}(x)$ such that $x(E) \leq x(F)$. The inequalities $x(E) \leq x(F) \leq x(E \cup F)$ and the V-property of x imply the existence of a set $F_1 \in$

$\mathcal{B}(\mathbb{R})$ such that $E \subseteq F_1 \subseteq E \cup F$ and $x(F_1) = x(F)$. Then $x(F) \ominus x(E) = x(F_1) \ominus x(E) = x(F_1 \backslash E) \in \mathcal{R}(x)$ and so $\mathcal{R}(x)$ is a sub-D-poset of $\mathcal{P}$.

The range of any observable is a set of pairwise compatible elements. If $\mathcal{R}(x)$ is a lattice, then it is a D-lattice of pairwise compatible elements, which is a Boolean D-poset. ■

We say that an observable x is *regular* if the inequality $x(E) \leq x(E)'$ implies $x(E) = 0_{\mathcal{P}}$.

Theorem 3.4. Let x be an observable in a D-poset $\mathcal{P}$. The following assertions are equivalent:

(i) x is regular.
(ii) $\mathcal{R}(x)$ is a Boolean subalgebra of $\mathcal{P}$.
(iii) If $A, B \in \mathcal{B}(\mathbb{R})$, $x(A) \leq x(B)$, then $x\,(A \cup B) = x(B)$ and $x\,(A \cap B) = x(A)$.

Proof. The implication (i) $\Rightarrow$ (ii) was proved by Dvurečenskij and Pulmannová [6].

The implication (ii) $\Rightarrow$ (iii) is true because $x(A \cup B) = x(A) \vee x(B)$ for any regular observable x.

Let (iii) hold. If $x(A) \leq x(A)' = x(A^c)$, then $x(A) = x(A \cap A^c) = x(\emptyset) = 0_{\mathcal{P}}$, which gives that x is regular. ■

Corollary 3.5. Let x be a regular observable in a D-poset $\mathcal{P}$. Then:

(i) $x(A) = x(B)$ if and only if $x(A \cup B) = x(A \cap B)$.
(ii) If $A \cap B = \emptyset$ and $x(A) = x(B)$, then $x(A) = 0_{\mathcal{P}} = x(B)$.
(iii) The observable x has the *V*-property on $\mathcal{R}(x)$.

4. JOINT OBSERVABLES IN D-POSETS

The notion of a joint observable is a quantum paraphrase of the notion of a random vector. Joint observables play an important role in solving some problems from the probability theory on non-Boolean structures. Results of probability theory on D-posets and MV-algebras can be found in refs. 11, 13, and 16.

If $\mathcal{P}$ is a D-poset and x, y are observables in $\mathcal{P}$, then by a *joint observable* of x and y we mean a σ-D-homomorphism w: $\mathcal{B}(\mathbb{R}^2) \to \mathcal{P}$ such that $w(E \times \mathbb{R}) = x(E)$ and $w(\mathbb{R} \times F) = y(F)$ for every $E, F \in \mathcal{B}(\mathbb{R})$.

We note that a joint observable in quantum logics exists only for compatible observables. [The observables x and y are compatible if $x(E) \leftrightarrow y(F)$ for every $E, F \in \mathcal{B}(\mathbb{R})$.] Indeed, the compatibility of x and y implies the existence of real-valued Borel-measurable functions f, g and an observable w such that $x = w \circ f^{-1}$ and $y = w \circ g^{-1}$. Let a mapping h: $\mathbb{R} \to \mathbb{R}^2$ be defined by the equality $h(t) = (f(t), g(t))$ for any $t \in \mathbb{R}$. Then a σ-homomorphism z from $\mathcal{B}(\mathbb{R}^2)$ into a quantum logic such that $z(A) = w(h^{-1}(A))$ for every $A \in \mathcal{B}(\mathbb{R}^2)$ is a joint observable of x and y.

We recall that a joint observable in quantum logics does not depend on the choice of functions f and g.

Now we will investigate joint observables in Boolean D-poset because here all elements are pairwise compatible.

First we present some basic notions.

A nonzero element a from a D-poset $\mathcal{P}$ is called an *atom* if the inequality $b \le a$ entails either $b = 0_{\mathcal{P}}$ or $b = a$. A D-poset is said to be *atomic* if for any nonzero element $b \in \mathcal{P}$ there exists an atom $a \in \mathcal{P}$ such that $a \le b$.

By a σ-*complete* D-poset we mean a D-poset $\mathcal{P}$ such that for a countable sequence $\{a_n\}_{n=1}^{\infty}$ the least upper bound $(\vee_{n=1}^{\infty} a_n)$ (and equivalently the greatest lower bound $\wedge_{n=1}^{\infty} a_n$) exists in $\mathcal{P}$.

Let $a \in \mathcal{P}$. We define

$$na := a_1 \oplus a_2 \oplus \cdots \oplus a_n$$

where $a_1 = a_2 = \cdots = a_n = a$ if the corresponding orthogonal sum exists in $\mathcal{P}$.

For any element $a \in \mathcal{P}$, $ord(a)$ is defined via

$$ord(a) = \sup\{n \ge 1: na \in \mathcal{P}\}$$

In a σ-complete Boolean D-poset one has $ord(a) < \infty$ for any nonzero element a [5].

The following theorem is analogous to the Cignoli theorem [4], according to which any atomic σ-complete MV-algebra (Boolean D-poset) can be expressed as a direct product of finite chains.

Theorem 3.6 [5]. Let $\{a_1, a_2, \ldots, a_n, \ldots\}$ be a countable system of all atoms in an atomic σ-complete Boolean D-poset $\mathcal{P}$ and $ord(a_n) = k_n$ for any $n = 1,2, \ldots$. Then the following assertions are true:

(i) $\oplus_{n=1}^{\infty}(k_n a_n) = 1_{\mathcal{P}}$.

(ii) For any $a \in \mathcal{P}$ there exist unique integers m_n, $m_n \in \{0,1, \ldots, k_n\}$, $n = 1, 2, \ldots$, such that

$$a = \bigoplus_{n=1}^{\infty}(m_n a_n)$$

Theorem 3.7. Let $\mathcal{P}$ be an atomic σ-complete Boolean D-poset with the countable set $\{a_1, a_2, \ldots, a_n, \ldots\}$ of all atoms such that $ord(a_n) = k_n$, $n = 1, 2, \ldots$. Then:

(1) There exists an observable w: $\mathcal{B}(\mathbb{R}) \to \mathcal{P}$ such that (i) $\mathcal{R}(w) =$

$\mathcal{P}$, (ii) $\sigma(w) = \cup_{n=1}^{\infty} W_n$, where $W_n \subseteq \mathbb{N}$, $W_i \cap W_j = \emptyset$ for $i \neq j$, and $w(W_n) = k_n a_n$ for any $n = 1, 2, \ldots$, and (iii) w has the V-property on $\mathcal{R}(w)$.

(2) If x is an observable such that $\mathcal{R}(x) = \mathcal{P}$ and x has the V-property on $\mathcal{R}(x)$, then there exists a bijection φ: $\sigma(w) \to \sigma(x)$.

(3) An observable $y := w \circ f^{-1}$ is regular if and only if the partial functions f/W_n are constant for any $n = 1, 2, \ldots$.

Proof. (1) Let W_n be a subset of the set of all positive integers $\mathbb{N}$ such that $|W_n| = k_n$ for $n = 1, 2, \ldots$, and $W_i \cap W_j = \emptyset$ for $i \neq j$, where $|W_n|$ denotes the cardinality of W_n.

We put $W = \cup_{n=1}^{\infty} W_n$ and denote by $\mathcal{S}$ a system of all subsets of W.

We now define a mapping h: $\mathcal{S} \to \mathcal{P}$ by

$$h(E) = \bigoplus_{n=1}^{\infty} m_n\, a_n \qquad \text{for any} \quad E \in \mathcal{S}$$

where $m_n = |E \cap W_n|$. Then h is a σ-D-homomorphism on $\mathcal{P}$.

Putting

$$w(E) = h(E \cap X) \qquad \text{for any} \quad E \in \mathcal{B}(\mathbb{R})$$

we obtain that w is an observable in $\mathcal{P}$ such that $\mathcal{R}(w) = \mathcal{P}$, $w(W_n) = k_n a_n$, $\sigma(w) = W$, and w has the V-property on $\mathcal{P}$. A detailed proof of (1) is given in ref. 5.

(2) From the assumptions (the observables w and x have the V-property) it follows that there exist Borel-measurable functions f and g such that $x = w \circ f^{-1}$ and $w = x \circ g^{-1}$. From the above we have $\sigma(w) = \cup_{n=1}^{\infty} W_n$. We show that the partial functions $\varphi_n = f/W_n$, $n = 1, 2, \ldots$, are bijections. The sets W_n are finite, so the images $f(W_n)$ are finite, too, and $|f(W_n)| \leq |W_n|$. We prove that $|f(W_n)| = |W_n|$ for any $n \in \mathbb{N}$.

Let $|f(W_n)| < |W_n|$. Then there exist $t_1, t_2 \in W_n$ such that $f(t_1) = f(t_2) = s \in \sigma(x)$ and

$$x(\{s\}) = w(f^{-1}(\{s\})) \geq w((\{t_1, t_2\}) = a_n \oplus a_n = 2a_n$$

For any $t \in W_n$

$$a_n = w(\{t\}) = x(g^{-1}(\{t\}))$$

From the inequalities $0_{\mathcal{P}} < x(g^{-1}(\{t\})) = a_n < 2a_n < x(\{s\})$ and the V-property of x it follows that there exists $A \in \mathcal{B}(\mathbb{R})$ such that $\emptyset \subsetneq A \subsetneq \{s\}$ and $x(A) = a_n$. Then either $A = \emptyset$ or $A = \{s\}$. Both eventualities lead to conflict. So, $|f(W_n)| = |W_n|$. Then the mappings φ_n: $t_{n_i} \mapsto f(t_{n_i})$, $i = 1, \ldots, k_n$, are bijections from W_n onto $f(W_n)$ for any $n = 1, 2, \ldots$. It suffices to put $\varphi := f/W$.

(3) Let f be a Borel-measurable real function and let the observable $y = w \circ f^{-1}$ be regular. Then $\mathcal{R}(y)$ is a Boolean subalgebra of $\mathcal{P}$.

Let $t_1, t_2 \in W_n$, $t_1 \neq t_2$, and $f(t_1) = s_1 \neq s_2 = f(t_2)$. Then $t_2 \notin f^{-1}(\{s_1\})$ and $|f^{-1}(\{s_1\}) \cap W_n| < |W_n|$, so $a_n \leq w(f^{-1}(\{s_1\}) \cap W_n) < k_n a_n$.

We have $a_n \leq w(f^{-1}(\{s_1\}) \cap W_n) \leq a_n'$ and also $(w(f^{-1}(\{s_1\}) \cap W_n))' \leq a_n'$; therefore, $1_{\mathcal{P}} = w(f^{-1}(\{s_1\}) \cap W_n) \vee (w(f^{-1}(\{s_1\}) \cap W_n))' \leq a_n'$, which gives $a_n = 0_{\mathcal{P}}$. This is the conflict with the assumption that a_n is an atom. Therefore f/W_n is the constant function for any $n \in \mathbb{N}$.

Now let $f(t) = s_n$ for any $t \in W_n$, $n = 1,2, \ldots$, and let $A \in \mathcal{B}(\mathbb{R})$, $A \neq \emptyset$, $y(A) > 0_{\mathcal{P}}$, $y(A) \leq (y(A))'$.

There are four possibilities:

(a) $f^{-1}(A) = \emptyset$.
(b) $f^{-1}(A) = \cup_{n=1}^{\infty} W_n = W$.
(c) $f^{-1}(A) = \cup_{n=j_1}^{j_p} W_n$, $j_1, \ldots, j_p \in \{1,2, \ldots\}$.
(d) $f^{-1}(A) = \cup_{n \in T} W_n$, $T = \mathbb{N} \backslash \{j_1, \ldots, j_p\}$.

In case (a) we have $y(A) = w(f^{-1}(A)) = w(\emptyset) = 0_{\mathcal{P}}$, which conflicts with the assumption $y(A) > 0_{\mathcal{P}}$.

In case (b) we have $y(A) = w(f^{-1}(A)) = w(W) = 1_{\mathcal{P}}$ and then $(y(A))' = 0_{\mathcal{P}}$, which conflicts with the assumption $y(A) \leq (y(A))'$.

Let (c) hold. Denote by $b_{j_i} = k_{j_i} a_{j_i}$, $i = 1, \ldots, p$. Then

$$\begin{aligned} y(A) = w(f^{-1}(A)) &= w\left(\bigcup_{n=j_1}^{j_p} W_n\right) \\ &= b_{j_1} \oplus b_{j_2} \oplus \ldots \oplus b_{j_p} \\ &= b_{j_1} \vee b_{j_2} \vee \ldots \vee b_{j_p} \end{aligned}$$

and

$$(y(A))' = b_{j_1}' \wedge b_{j_2}' \wedge \ldots \wedge b_{j_p}'$$

The inequalities

$$k_{j_1} a_{j_1} = b_{j_1} \leq y(A) \leq (y(A))' \leq (b_{j_1})' \leq (a_{j_1})'$$

imply that the orthogonal sum $(k_{j_1} a_{j_1}) \oplus a_{j_1}$ exists and

$$(k_{j_1} a_{j_1}) \oplus a_{j_1} = (k_{j_1} + 1) a_{j_1}$$

which conflicts with the assumption $ord(a_{j_1}) = k_{j_1}$.

Similarly we get conflict in case (d).

Therefore, for any $A \in \mathcal{B}(\mathbb{R})$ such that $y(A) \leq (y(A))'$ we have $y(A) = 0_{\mathcal{P}}$, which gives that the observable y is regular. ■

Theorem 3.8. For any two regular observables in an atomic σ-complete Boolean D-poset with the countable set $\{a_1, a_2, \ldots\}$ of all atoms there exists a joint observable.

Proof. By Theorem 3.7 there exists an observable w such that $\mathcal{R}(w) = \mathcal{P}$, $\sigma(w) = W = \cup_{n=1}^{\infty} W_n$ and w has the V-property on $\mathcal{P}$. Then the observables x and y are w-representable, that is, $x = w \circ f^{-1}$ and $y = w \circ g^{-1}$ and the functions f/W_n, g/W_n, $n = 1, 2, \ldots$, are constant. It is easy to verify that the functions f/W and g/W are defined uniquely. According to classical quantum logics theory, we define a mapping h: $\mathbb{R} \to \mathbb{R}^2$, $h(t) = (f(t), g(t))$ for any $t \in \mathbb{R}$. Then a mapping z from $\mathcal{B}(\mathbb{R}^2)$ into $\mathcal{P}$ such that $z(A) = w(h^{-1}(A))$ for every $A \in \mathcal{B}(\mathbb{R}^2)$ is a joint observable of x and y. ■

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